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Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems ☆

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ABSTRACT

Some critical point theorems without the compactness assumptions are obtained by the reduction method, the perturbation argument and the least action principle. As applications, some existence results of periodic solutions are obtained for nonautonomous second order Hamiltonian systems, which unify and generalize some recent corresponding results.

1. Introduction and main results

Most of critical point theorems are based on the compactness assumptions (see [2,6,10,12,14,29,33,35,48,57] and their references). Some critical point theorems are without the compactness assumptions (see [8,9,17,19,25,34] and their references). A.C. Lazer, E.M. Landesman and D.R. Meyers [19] proved the following critical point theorem without the compactness assumptions.

Theorem A. *Suppose that V and W are two closed subspaces of a Hilbertian space H such that $H = V \oplus W$, V finite-dimensional, V and W not necessarily orthogonal. Assume that $\varphi \in C^2(H, R)$ such that*

$$\langle \varphi''(u)v, v\rangle \leqslant -m_1\|v\|^2$$

and

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$$\langle \varphi''(u)w, w \rangle \geqslant m_2 \|w\|^2$$

for all $u \in H$, $v \in V$, $w \in W$ and some positive constants m_1 and m_2. Then there exists a unique $u_0 \in H$ such that $\varphi'(u_0) = 0$, and u_0 satisfies

$$\varphi(u_0) = \max_{v \in V} \min_{w \in W} \varphi(v + w).$$

On one hand, A. Castro and A.C. Lazer [9] weaken the condition that $\langle \varphi''(u)v, v \rangle \leqslant -m_1 \|v\|^2$ to that $\varphi(v) \to -\infty$ as $\|v\| \to \infty$, $v \in V$. This result is a corollary of the following generalization of von Neumann minimax theorem (see e.g. [3, Theorem 8 at p. 319]):

Theorem B *(von Neumann minimax theorem). Suppose that M and N are two convex subsets of linear spaces, supplied with topologies. Assume that $\varphi : M \times N \to R$, $\varphi(v, w)$ is convex and lower semi-continuous in v for all $w \in N$, and there exists $w_0 \in N$ such that the subset $\{v \in M \mid \varphi(v, w_0) \leqslant c\}$ is compact for all $c \in R$, and $\varphi(v, w)$ is concave and upper semi-continuous in w for all $v \in M$, and there exists $v_0 \in M$ such that the subset $\{w \in M \mid \varphi(v_0, w) \leqslant c\}$ is compact for all $c \in R$. Then there exists a saddle point $(\bar{v}, \bar{w}) \in M \times N$, that is,*

$$\varphi(\bar{v}, w) \leqslant \varphi(\bar{v}, \bar{w}) \leqslant \varphi(v, \bar{w}),$$

for all $(v, w) \in M \times N$.

On the other hand, V.L. Shapiro [34] and R.F. Manasevich [25] generalize Theorem A by dropping the condition that V is finite-dimensional. See P.W. Bates and I. Ekeland [4] and R.F. Manasevich [24] for others related research.

In this paper, using the reduction method, the perturbation argument and the least action principle, we obtain the following critical point theorem which generalizes Theorem A.

Theorem 1.1. *Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$ and $\varphi(\cdot, w) : V \to R$ is convex for all $w \in W$, that is,*

$$\varphi\big(\lambda v_1 + (1 - \lambda)v_2, w\big) \leqslant \lambda\varphi(v_1, w) + (1 - \lambda)\varphi(v_2, w)$$

for all $\lambda \in [0, 1]$ and $v_1, v_2 \in V$, $w \in W$, and φ' is weakly continuous. Assume that

$$\varphi(0, w) \to -\infty \tag{1}$$

as $\|w\| \to \infty$ and, for every $M > 0$,

$$\varphi(v, w) \to +\infty \tag{2}$$

as $\|v\| \to \infty$ uniformly for $\|w\| \leqslant M$. Then φ has at least one critical point.

Remark 1.1. Theorem 1.1 generalizes Theorem A in three aspects. At first Theorem 1.1 requires the spaces being reflexive Banach spaces instead of Hilbert spaces; secondly Theorem 1.1 requires the functionals being C^1 instead of C^2; more important is that Theorem 1.1 requires weaker convexity of the functionals. With variant methods, some results similar to Theorem 1.1 are obtained in [8] and [17]. They assume that V is finite-dimensional or

$$\varphi = q + \psi$$

in the Hilbert space case, and φ is strict quasi-concave in the Banach space case.

Remark 1.2. Theorem 1.1 is a useful complement of von Neumann minimax theorem (see Theorem B), for we do not need the condition that $\varphi(v,w)$ is concave in w for all $v \in V$ in Theorem 1.1.

As applications, we consider the nonautonomous second-order Hamiltonian systems

$$\begin{cases} -\ddot{u}(t) = \nabla F\big(t, u(t)\big) \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{3}$$

where $T > 0$ and $F : [0,T] \times R^N \to R$, $(t,x) \to F(t,x)$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0,T;R^+)$ such that

$$\big|F(t,x)\big| + \big|\nabla F(t,x)\big| \leqslant a\big(|x|\big)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

Many solvability conditions for problem (3) are obtained, such as: the coercivity condition (see [7, 27,38,44] and their references); the convexity conditions (see [26,37,52,40] and their references); the sublinear nonlinearity conditions (see [16,27,39,43] and their references); the subquadratic potential conditions (see [18,31,42,43] and their references); the superquadratic potential conditions (see [5,15, 20,23,30,32,45,46] and their references); the periodicity conditions (see [11,21,28,41,49,51] and their references) and the even type potential condition (see [22,50,53] and their references).

By the dual least action principle and the perturbation technique J. Mawhin and M. Willem [27] obtain the following theorem.

Theorem C. *(See [27].) Suppose that $F(t,x)$ is measurable in t for each $x \in R^N$, and continuously differentiable and convex in x for a.e. $t \in [0,T]$. Assume that the following conditions are satisfied*:

(A_1) *There exists $l \in L^4(0,T;R^N)$ such that*

$$\big(l(t), x\big) \leqslant F(t,x)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

(A_2) *There exist $\alpha \in]0, \omega^2[$ and $\gamma \in L^2(0,T;R^+)$ such that*

$$F(t,x) \leqslant \frac{1}{2}\alpha|x|^2 + \gamma(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, where $\omega = 2\pi/T$.

(A_3)

$$\int_0^T F(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty,\ x \in R^N.$$

Then problem (3) *has at least one solution in H_T^1.*

Then applying the dual least action principle and the perturbation technique to semilinear equation on reflexive Banach space, as a corollary C.-L. Tang [36] slightly generalizes Theorem C by relaxing the integrability of l and γ, and obtains the following theorem.

Theorem D. *(See [36].) Suppose that $F(t,x)$ is measurable in t for each $x \in R^N$, and continuously differentiable and convex in x for a.e. $t \in [0,T]$. Assume that* (A_3) *holds and the following conditions are satisfied*:

(A_1') *There exists $l \in L^2(0,T;R^N)$ such that*

$$\bigl(l(t),x\bigr) \leqslant F(t,x)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

(A_2') *There exist $\alpha \in]0,\omega^2[$ and $\gamma \in L^1(0,T;R^+)$ such that*

$$F(t,x) \leqslant \frac{1}{2}\alpha|x|^2 + \gamma(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

Then problem (3) *has at least one solution.*

Remark 1.3. Assumption (A) holds for functions F in Theorems C and D. In fact, by (A_1') and (A_2') we have

$$\begin{aligned}\bigl|F(t,x)\bigr| &\leqslant \bigl|l(t)\bigr||x| + \frac{1}{2}\alpha|x|^2 + \gamma(t)\\ &\leqslant (\alpha+1)\bigl(|x|+1\bigr)^2\bigl(1+\bigl|l(t)\bigr|+\gamma(t)\bigr)\end{aligned}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$. It follows from the convexity of $F(t,\cdot)$ that

$$F(t,y) \geqslant F(t,x) + \bigl(\nabla F(t,x), y-x\bigr)$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$. Let $z = \nabla F(t,x)$ and $y = x + z/|z|$ for $z \neq 0$. Then one has

$$\begin{aligned}\bigl|\nabla F(t,x)\bigr| &= |z|\\ &= \bigl(\nabla F(t,x), y-x\bigr)\\ &\leqslant F(t,y) - F(t,x)\\ &\leqslant \frac{1}{2}\alpha|y|^2 + \gamma(t) - \bigl(l(t),x\bigr)\\ &\leqslant \frac{1}{2}\alpha\bigl(|x|+1\bigr)^2 + \gamma(t) + \bigl|l(t)\bigr||x|\\ &\leqslant (\alpha+1)\bigl(|x|+1\bigr)^2\bigl(1+\bigl|l(t)\bigr|+\gamma(t)\bigr)\end{aligned}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Hence assumption (A) holds with $a(s) = 2(\alpha+1)(s+1)^2$, $b(t) = 1 + |l(t)| + \gamma(t)$.

In this paper, using our abstract critical point theorem we obtain the following theorem which generalizes the results mentioned above.

Theorem 1.2. *Suppose that assumption* (A) *holds and $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Assume that* (A_3) *holds and*:

(A_4)

$$F(t,x) - \frac{1}{2}\omega^2|x|^2 \to -\infty \quad as\ |x| \to \infty, \tag{4}$$

for a.e. $t \in [0, T]$.

Then problem (3) *has at least one solution in* H_T^1.

Remark 1.4. Theorem 1.2 generalizes Theorem D. Consequently Theorem 1.2 also generalizes Theorem C. It is obvious that Theorem 1.2 implies Theorem D. There are functions F satisfying our Theorem 1.2 and not satisfying Theorems C and D. For example, let

$$F(t,x) = \frac{1}{2}\omega^2|x|^2 - \frac{1}{2}\omega^2\left(1+|x|^2\right)^{\frac{3}{4}} + \left(l(t), x\right),$$

where $l \in L^\infty(0, T; R^N)$. By Remark 1.3, assumption (A) holds. Clearly (A_3) and (4) hold, and F is convex in x for a.e. $t \in [0, T]$ because

$$f(x) \triangleq g\left(h(x)\right)$$

is convex by the fact that

$$g(s) \triangleq s - (1+s)^{\frac{3}{4}}, \quad s \geqslant 0,$$

is convex and increasing and

$$h(x) \triangleq |x|^2, \quad x \in R^N,$$

is convex. Thus F satisfies the conditions of our Theorem 1.2. But obviously F does not satisfy the conditions of Theorem D, for (A_2) does not hold.

Remark 1.5. It seems that one cannot obtain our Theorem 1.2 with the methods used in [27] or [36].

Recently, Zhao and Wu [55] consider a class of unnecessarily convex Hamiltonian systems. Using the reduction method they obtain the following theorem.

Theorem E. *(See [55].) Suppose that assumption* (A) *holds and there exists a function* $\mu \in L^1(0, T; R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0, T]$. *Assume that there exist* $p \in L^1(0, T; R^N)$ *and* $g \in L^1(0, T; R^+)$ *such that*

$$F(t,x) \leqslant \left(p(t), x\right) + g(t) \tag{5}$$

for all $x \in R^N$ *and a.e.* $t \in [0, T]$. *Then problem* (3) *has at least one solution in* H_T^1.

Subsequently, Wu [47] proves the following theorem.

Theorem F. *(See [47].) Suppose that assumption* (A) *holds and there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in x for a.e.* $t \in [0,T]$. *Assume that there exist* $f, g \in L^1(0,T;R^+)$ *with* $\int_0^T f(t)\,dt < 12/T$ *such that*

$$\big|\nabla F(t,x)\big| \leqslant f(t)|x| + g(t) \tag{6}$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$. *Then problem* (3) *has at least one solution in* H_T^1.

In this paper, using our abstract critical point theorem we obtain the following theorem which unifies and generalizes Theorems E and F.

Theorem 1.3. *Suppose that assumption* (A) *holds and there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in x for a.e.* $t \in [0,T]$. *Assume that there exist* $\alpha \in L^1(0,T;R^+)$ *with* $\int_0^T \alpha(t)\,dt < 12/T$ *and* $\gamma \in L^1(0,T;R^+)$ *such that*

$$F(t,x) \leqslant \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{7}$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$. *Then problem* (3) *has at least one solution in* H_T^1.

Remark 1.6. Theorem 1.3 unifies and generalizes Theorems E and F. In fact, on one hand, it follows from (5) that

$$\begin{aligned} F(t,x) &\leqslant \big(p(t),x\big) + g(t) \\ &\leqslant \big|p(t)\big||x| + g(t) \\ &\leqslant \frac{1}{2}\big|p(t)\big|\left(\frac{12}{T(\|p\|_{L^1}+1)}|x|^2 + \frac{T(\|p\|_{L^1}+1)}{12}\right) + g(t) \end{aligned}$$

which is just (7) with $\alpha(t) = \frac{12|p(t)|}{T(\|p\|_{L^1}+1)}$ and $\gamma(t) = \frac{T(\|p\|_{L^1}+1)}{24}|p(t)| + g(t)$. Hence Theorem 1.3 implies Theorem E. On the other hand, by (6) and assumption (A) we have

$$\begin{aligned} F(t,x) &= \int_0^1 \big(\nabla F(t,sx),x\big)\,ds + F(t,0) \\ &\leqslant \int_0^1 \big(f(t)|sx| + g(t)\big)|x|\,ds + F(t,0) \\ &= \frac{1}{2}f(t)|x|^2 + g(t)|x| + F(t,0) \\ &\leqslant \frac{1}{2}f(t)|x|^2 + \frac{1}{2}g(t)\left(\frac{12 - T\|f\|_{L^1}}{T(\|g\|_{L^1}+1)}|x|^2 + \frac{T(\|g\|_{L^1}+1)}{12 - T\|f\|_{L^1}}\right) + a(0)b(t) \end{aligned}$$

which is just (7) with

$$\alpha(t) = f(t) + \frac{12 - T\|f\|_{L^1}}{T(\|g\|_{L^1}+1)}g(t)$$

and

$$\gamma(t) = \frac{T(\|g\|_{L^1} + 1)}{2(12 - T\|f\|_{L^1})} g(t) + a(0)b(t).$$

Thus Theorem 1.3 implies Theorem F. There are functions F satisfying our Theorem 1.3 and not satisfying Theorems E and F. For example, let

$$F(t,x) = \frac{1}{2}\mu(t)|x|^2 + \big(p(t), x\big)$$

where $\mu \in L^\infty(0,T;R)$ with

$$\frac{6}{T} \leqslant \int_0^T \mu^-(t)\,dt < \int_0^T \mu^+(t)\,dt < \frac{12}{T},$$

$\mu^\pm(t) = \max\{\pm\mu(t), 0\}$ and $p \in L^1(0,T;R^N)$. Then one has

$$\begin{aligned}
F(t,x) &= \frac{1}{2}\mu(t)|x|^2 + \big(p(t), x\big) \\
&\leqslant \frac{1}{2}\mu^+(t)|x|^2 + \big|p(t)\big||x| \\
&\leqslant \frac{1}{2}\mu^+(t)|x|^2 + \frac{1}{2}\big|p(t)\big|\left(\frac{12 - T\|\mu^+\|_{L^1}}{T(\|p\|_{L^1} + 1)}|x|^2 + \frac{T(\|p\|_{L^1} + 1)}{12 - T\|\mu^+\|_{L^1}}\right)
\end{aligned}$$

which is just (7) with

$$\alpha(t) = \mu^+(t) + \frac{12 - T\|\mu^+\|_{L^1}}{T(\|p\|_{L^1} + 1)}\big|p(t)\big|$$

and

$$\gamma(t) = \frac{T(\|p\|_{L^1} + 1)}{2(12 - T\|\mu^+\|_{L^1})}\big|p(t)\big|.$$

Hence F satisfies our Theorem 4.1. But F does not satisfy the conditions of Theorem E or Theorem F, for $\int_0^T |\mu(t)|\,dt > 12/T$ and (6) does not hold.

Zhao and Wu [55] also obtain the following theorem by using the idea of reduction. Then Wu [47] gives a simple proof with the reduced methods.

Theorem G. *(See [55,47].) Suppose that assumption* (A) *holds and there exists $k \in L^1(0,T;R^+)$ with $\int_0^T k(t)\,dt < 12/T$ such that*

$$\big|\nabla F(t,x) - \nabla F(t,y)\big| \leqslant k(t)|x - y| \tag{8}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$. Assume that (A_3) *is satisfied. Then problem* (3) *has at least one solution in H_T^1.*

In order to weaken condition (8), Zhao and Wu [56] add some other conditions and obtain the following theorem.

Theorem H. *(See [56].) Suppose that assumption* (A) *holds and there exists* $k \in L^1(0,T;R^+)$ *with* $0 < \int_0^T k(t)\,dt < 12/T$ *such that* $-F(t,x) + \frac{1}{2}k(t)|x|^2$ *is convex in x for a.e.* $t \in [0,T]$*. Assume that* (A_3) *is satisfied,* $\nabla F(t,0) = 0$*,*

$$F\big(t, \lambda(x+y)\big) \geqslant \mu\big(F(t,x) + F(t,y)\big)$$

for some λ, μ *and all* $x, y \in R^N$*, and there exist* $f, g \in L^1(0,T;R^+)$ *such that*

$$F(t,x) \leqslant f(t)|x|^2 + g(t)$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$*. Then problem* (3) *has at least one solution in* H_T^1*.*

However Mawhin and Willem [27] obtain the following results with the least action principle.

Theorem I. *(See [56].) Suppose that assumption* (A) *holds and* $-F(t,x)$ *is convex in x for a.e.* $t \in [0,T]$*. Assume that* (A_3) *is satisfied. Then problem* (3) *has at least one solution in* H_T^1*.*

In this paper, using our abstract critical point theorem we obtain the following theorem which generalizes Theorems G, H and I.

Theorem 1.4. *Suppose that assumption* (A) *holds and there exists* $k \in L^1(0,T;R^+)$ *with* $\int_0^T k(t)\,dt < 12/T$ *such that* $-F(t,x) + \frac{1}{2}k(t)|x|^2$ *is convex in x for a.e.* $t \in [0,T]$*. Assume that* (A_3) *is satisfied. Then problem* (3) *has at least one solution in* H_T^1*.*

Remark 1.7. Theorem 1.4 unifies and generalizes Theorems G, H and I. In fact, on one hand, assume that (8) holds. Then one has

$$\begin{aligned}\big(\nabla\big(-F(t,x)\big) - \nabla\big(-F(t,y)\big), x-y\big) &\geqslant -\big|\nabla F(t,x) - \nabla F(t,y)\big||x-y| \\ &\geqslant -k(t)|x-y|^2\end{aligned}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$, which implies that $-F(t,x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0,T]$. Hence Theorem 1.4 implies Theorem G. On the other hand, Theorem 1.4 completely drops conditions that $\int_0^T k(t)\,dt > 0$, $\nabla F(t,0) = 0$,

$$F\big(t, \lambda(x+y)\big) \geqslant \mu\big(F(t,x) + F(t,y)\big)$$

for some λ, μ and all $x, y \in R^N$, and there exist $f, g \in L^1(0,T;R^+)$ such that $F(t,x) \leqslant f(t)|x|^2 + g(t)$ for all $x \in R^N$ and a.e. $t \in [0,T]$ in Theorem H. Theorem I is just Theorem 1.4 with $k = 0$. There are functions F satisfying our Theorem 1.4 and not satisfying Theorems G, H or I. For example, let

$$F(t,x) = \frac{1}{2}\mu(t)|x|^2 + \big(p(t), x\big)$$

where $\mu \in L^\infty(0,T;R)$ with

$$\frac{6}{T} \leqslant \int_0^T \mu^-(t)\,dt < \int_0^T \mu^+(t)\,dt < \frac{12}{T},$$

and $p \in L^1(0,T;R^N) \setminus \{0\}$. Then F satisfies our Theorem 1.4 with $k = \mu^+$. But F does not satisfy the conditions of Theorem G, Theorem H or Theorem I, for $\int_0^T |\mu(t)|\,dt > 12/T$, $\nabla F(t,0) = p(t) \neq 0$ and $F(t,x)$ is not convex in x for $t \in [0,T]$ with $\mu^-(t) > 0$.

This paper is organized as follows. In Section 2, the proof of Theorems 1.1 and 2.1 will be given. In Section 3, we intend to give the proof of Theorem 1.2, the generalization of Theorem 1.2 and some other corresponding results. The proof of Theorem 1.3 and some other corresponding results will be given in Section 4. In the last section, we manage to give the proof of Theorem 1.4 and some other corresponding results.

2. Some critical point theorems

In this section, we shall give the proof of Theorem 1.1 by the reduction method, the perturbation argument and the least action principle. For convenience to quote, recall the well-known least action principle. It can be found for example in [27].

The least action principle Suppose that V is a reflexive Banach space and $\varphi : V \to R$ is weakly lower semi-continuous. Assume that φ is coercive, that is,

$$\varphi(v) \to +\infty \quad \text{as } \|v\| \to \infty.$$

Then φ has at least one minimum.

Before giving the proof of Theorem 1.1, we prove the following lemma.

Lemma 2.1. *Suppose that V is a reflexive Banach space and W is a Banach space, $\varphi \in C^1(V \times W, R)$. Assume that there exists $\mu > 0$ such that $D_1\varphi(\cdot, w) : V \to V'$ is μ-monotone for all $w \in W$, that is,*

$$\big\langle D_1\varphi(v_1,w) - D_1\varphi(v_2,w), v_1 - v_2\big\rangle \geqslant \mu\|v_1 - v_2\|^2$$

for all $v_1, v_2 \in V$ and $w \in W$. Then there exists a map $\theta \in C(W,V)$ such that $\theta(w)$ is the unique minimum of $\varphi(\cdot, w)$ for all $w \in W$ and the functional ψ, given by

$$\psi(w) = \varphi\big(\theta(w), w\big) = \inf_{v \in V} \varphi(v,w),$$

is continuously differentiable and

$$\psi'(w) = D_2\varphi\big(\theta(w), w\big)$$

for all $w \in W$. Moreover, $(\theta(w), w)$ is a critical point of φ if and only if w is a critical point of ψ.

Proof. Because $D_1\varphi(\cdot, w) : V \to V'$ is μ-monotone for all $w \in W$, we know that $\varphi(v,w) - \frac{1}{2}\mu\|v\|^2$ is convex for every $w \in W$. By the definition of sub-differentials, one has

$$\varphi(v,w) - \frac{1}{2}\mu\|v\|^2 \geqslant \varphi(0,w) + \big\langle D_1\varphi(0,w), v\big\rangle$$

for all $v \in V$ and $w \in W$. Hence one obtains

$$\varphi(v,w) \geqslant \frac{1}{2}\mu\|v\|^2 - \|D_1\varphi(0,w)\|\|v\| + \varphi(0,w)$$

for all $v \in V$ and $w \in W$, which implies that $\varphi(\cdot,w)$ is coercive for every $w \in W$. Now the weak lower semi-continuity of $\varphi(\cdot,w)$ follows from the continuity and Mazur Theorem. By the least action principle, φ has at least one minimum. The uniqueness of the minimum of φ follows from the μ-monotonity of $D_1\varphi(\cdot,w)$. Denote the unique minimum of $\varphi(\cdot,w)$ by $\theta(w)$ for every $w \in W$ and let

$$\psi(w) = \varphi\big(\theta(w),w\big) = \inf_{v\in V}\varphi(v,w)$$

for all $w \in W$.

It follows from the definition of θ that

$$D_1\varphi\big(\theta(w),w\big) = 0 \tag{9}$$

for all $w \in W$. By (9) and the μ-monotonity of $D_1\varphi(\cdot,w)$, we have

$$\begin{aligned}\mu\big\|\theta(w)-\theta(w_0)\big\|^2 &\leqslant \big\langle D_1\varphi\big(\theta(w),w\big) - D_1\varphi\big(\theta(w_0),w\big), \theta(w)-\theta(w_0)\big\rangle \\ &= \big\langle D_1\varphi\big(\theta(w_0),w_0\big) - D_1\varphi\big(\theta(w_0),w\big), \theta(w)-\theta(w_0)\big\rangle \\ &\leqslant \big\|D_1\varphi\big(\theta(w_0),w\big) - D_1\varphi\big(\theta(w_0),w_0\big)\big\|\big\|\theta(w)-\theta(w_0)\big\|\end{aligned}$$

for all $w, w_0 \in W$, that is,

$$\mu\big\|\theta(w)-\theta(w_0)\big\| \leqslant \big\|D_1\varphi\big(\theta(w_0),w\big) - D_1\varphi\big(\theta(w_0),w_0\big)\big\|$$

for all $w, w_0 \in W$. Now the continuity of θ follows from that of $D_1\varphi(\theta(w_0),\cdot)$. By the definition of θ and $D_2\varphi(\theta(w),w)$ one has

$$\begin{aligned}&\psi(w+h)-\psi(w)-\big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad= \varphi\big(\theta(w+h),w+h\big) - \varphi\big(\theta(w),w\big) - \big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad\leqslant \varphi\big(\theta(w),w+h\big) - \varphi\big(\theta(w),w\big) - \big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad= o\big(\|h\|\big)\end{aligned}$$

for all $w \in W$. It follows from the definition of θ and the continuity of $D_2\varphi(\theta(w),w)$ that

$$\begin{aligned}&\psi(w+h)-\psi(w)-\big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad= \varphi\big(\theta(w+h),w+h\big) - \varphi\big(\theta(w),w\big) - \big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad\geqslant \varphi\big(\theta(w+h),w+h\big) - \varphi\big(\theta(w+h),w\big) - \big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle \\ &\quad= \int_0^1 \big\langle D_2\varphi\big(\theta(w+h),w+th\big) - D_2\varphi\big(\theta(w),w\big),h\big\rangle\,dt \\ &\quad= o\big(\|h\|\big)\end{aligned}$$

for all $w \in W$. Hence we obtain

$$\psi(w+h)-\psi(w)-\big\langle D_2\varphi\big(\theta(w),w\big),h\big\rangle=o\big(\|h\|\big)$$

for all $w\in W$, that is

$$\psi'(w)=D_2\varphi\big(\theta(w),w\big)$$

for all $w\in W$. Therefore $\psi(w)$ is continuously differentiable. □

Remark 2.1. When V and W are Hilbert spaces, Lemma 2.1 is a corollary of Theorem 2.3 in Amann [1].

Now we can obtain the following critical point theorems by Lemma 2.1 and the least action principle.

Theorem 2.1. *Suppose that V and W are reflexive Banach spaces, $\varphi\in C^1(V\times W,R)$ and there exists $\mu>0$ such that $D_1\varphi(\cdot,w):V\to V'$ is μ-monotone for all $w\in W$. Assume that $\varphi(v,\cdot)$ is weakly upper semi-continuous for all $v\in V$ and*

$$\varphi(0,w)\to-\infty$$

as $\|w\|\to\infty$. Then φ has at least one critical point (v_0,w_0) such that

$$\varphi(v_0,w_0)=\inf_{v\in V}\varphi(v,w_0)=\sup_{w\in W}\inf_{v\in V}\varphi(v,w).$$

Proof. It follows that $-\varphi(v,\cdot)$ is weakly lower semi-continuous for all $v\in V$. By the properties of weak lower semi-continuity,

$$-\psi(w)=\sup_{v\in V}\big\{-\varphi(v,w)\big\},$$

is weakly lower semi-continuous. Moreover, by

$$-\psi(w)\geqslant-\varphi(0,w)$$

we know that $-\psi$ is coercive. It follows from the least action principle that $-\psi$ has a minimum w_0. By Lemma 2.1, $(\theta(w_0),w_0)$ is a critical point of φ. Let $v_0=\theta(w_0)$. Then (v_0,w_0) is a critical point of φ. Moreover one has

$$\varphi(v_0,w_0)=\varphi\big(\theta(w_0),w_0\big)=\psi(w_0)=\inf_{v\in V}\varphi(v,w_0)$$

and

$$\varphi(v_0,w_0)=\varphi\big(\theta(w_0),w_0\big)=\psi(w_0)=\sup_{w\in W}\psi(w)=\sup_{w\in W}\inf_{v\in V}\varphi(v,w).\qquad\square$$

At last in this section we prove Theorem 1.1 by the perturbation argument and Theorem 2.1.

Proof of Theorem 1.1. It is well known that every reflexive Banach space has an equivalent norm which is Frechet differentiable (see [13]). Without loss of generality we may assume that the norm of V is Frechet differentiable. For every positive integer n, define

$$\varphi_n(v,w)=\varphi(v,w)+\frac{1}{2n}\|v\|^2.$$

Then $D_1\varphi_n(\cdot, w): V \to V'$ is $\frac{1}{n}$-monotone for all $w \in W$, $\varphi_n(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$ and

$$\varphi_n(0, w) = \varphi(0, w) \to -\infty$$

as $\|w\| \to \infty$.

By Theorem 2.1, φ_n has at least one critical point (v_n, w_n) such that

$$\varphi_n(v_n, w_n) = \inf_{v\in V} \varphi_n(v, w_n) = \sup_{w\in W} \inf_{v\in V} \varphi_n(v, w). \tag{10}$$

It follows from (1), (10) and (2) that

$$\begin{aligned}
\varphi(0, w_n) &= \varphi_n(0, w_n) \\
&\geqslant \varphi_n(v_n, w_n) \\
&\geqslant \inf_{v\in V} \varphi_n(v, 0) \\
&\geqslant \inf_{v\in V} \varphi(v, 0) \\
&> -\infty
\end{aligned}$$

for all $n \in N$, which implies that $\{w_n\}$ is bounded by (1). Moreover, $\sup_n \varphi(0, w_n) < +\infty$. If not, w_n has a subsequence, still denoted by w_n, such that

$$\varphi(0, w_n) \to +\infty \quad \text{as } n \to \infty. \tag{11}$$

By the reflexivity of W, there exists a subsequence, still denoted by w_n, and $w_0 \in W$ such that $w_n \to w_0$ weakly as $n \to \infty$. By the weak upper semi-continuity of $\varphi(0, \cdot)$ we have

$$\limsup_{n\to\infty} \varphi(0, w_n) \leqslant \varphi(0, w_0)$$

which contradicts (11). Now, the boundedness of $\{v_n\}$ follows from (2) and the inequality

$$\begin{aligned}
+\infty &> \sup \varphi(0, w_n) \\
&\geqslant \varphi(0, w_n) \\
&= \varphi_n(0, w_n) \\
&\geqslant \varphi_n(v_n, w_n) \\
&\geqslant \varphi(v_n, w_n).
\end{aligned}$$

Hence $\{(v_n, w_n)\}$ is bounded.

By the reflexivity of $V \times W$, $\{(v_n, w_n)\}$ has a subsequence, still denoted by $\{(v_n, w_n)\}$, which weakly converges to some $(v_0, w_0) \in V \times W$. Note that

$$\left\langle \varphi_n'(v_n, w_n), (h, k)\right\rangle = \left\langle \varphi'(v_n, w_n), (h, k)\right\rangle + \frac{1}{n}\langle v_n, h\rangle = 0, \quad \forall h \in V,\ k \in W,$$

and the weak continuity of φ', we have

$$\varphi'(v_0, w_0) = 0.$$

Therefore, (v_0, w_0) is a critical point of φ. $\square$

3. Periodic solutions of second-order nonautonomous subquadratic convex Hamiltonian systems

In this section, we shall give the proof of Theorem 1.2, the generalization of Theorem 1.2 and some other corresponding results. Theorem 1.2 is a straight corollary of the following Theorem 3.1.

Theorem 3.1. *Suppose that assumption* (A) *holds and $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Assume that* (A_3) *holds and*:

(A_5) *There exists $\gamma \in L^1(0,T;R^+)$ such that*

$$F(t,x) \leqslant \frac{1}{2}\omega^2|x|^2 + \gamma(t) \tag{12}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and

$$\operatorname{meas}\left\{t \in [0,T] \;\middle|\; F(t,x) - \frac{1}{2}\omega^2|x|^2 \to -\infty \text{ as } |x| \to \infty\right\} > 0. \tag{13}$$

Then problem (3) *has at least one solution in H_T^1.*

Remark 3.1. Theorem 3.1 generalizes Theorem 1.2. Obviously Theorem 1.2 follows from Theorem 3.1. In fact, it follows from (4) that there exists a positive constant M such that

$$F(t,x) - \frac{1}{2}\omega^2|x|^2 \leqslant 0$$

for all $x \in R^N$ with $|x| \geqslant M$ and a.e. $t \in [0,T]$. By assumption (A) one has

$$F(t,x) - \frac{1}{2}\omega^2|x|^2 \leqslant \Bigl(\max_{0 \leqslant s \leqslant M} a(s)\Bigr) b(t) \triangleq \gamma(t)$$

for all $x \in R^N$ with $|x| \leqslant M$ and a.e. $t \in [0,T]$. Hence (12) follows from (4) and assumption (A). Moreover there are functions F satisfying the conditions of our Theorem 3.1 and not satisfying the conditions of Theorem 1.2. For example, let

$$F(t,x) = \frac{1}{2}\omega^2|x|^2 - \frac{1}{2}\omega^2 \sin^2 \omega t \bigl(1 + |x|^2\bigr)^{\frac{3}{4}} + \bigl(l(t), x\bigr),$$

where $l \in L^3(0,T;R^N) \setminus L^\infty(0,T;R^N)$. Then by Young's inequality, one has

$$\begin{aligned}
-\frac{1}{2}\omega^2\bigl(1 + |x|^2\bigr)^{\frac{3}{4}} + \bigl(l(t), x\bigr) &\leqslant -\frac{1}{2}\omega^2|x|^{\frac{3}{2}} + \bigl|l(t)\bigr||x| \\
&\leqslant -\frac{1}{2}\omega^2|x|^{\frac{3}{2}} + \frac{2}{3}\left(\frac{1}{2}\omega^2|x|^{\frac{3}{2}}\right) + \frac{1}{3}\bigl(4\omega^{-4}\bigl|l(t)\bigr|^3\bigr) \\
&\leqslant \frac{4}{3}\omega^{-4}\bigl|l(t)\bigr|^3
\end{aligned}$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. Hence F satisfies (12) with $\gamma(t) = \frac{4}{3}\omega^{-4}|l(t)|^3$. Clearly (13) holds and F is convex in x for a.e. $t \in [0, T]$ in a way similar to Remark 1.4. Thus F satisfies the conditions of our Theorem 3.1. But obviously F does not satisfy the conditions of Theorem 1.2, for (A_4) does not hold.

Corollary 3.1. *Suppose that assumption* (A) *holds and $F(t, x)$ is convex in x for a.e. $t \in [0, T]$. Assume that* (A_3) *holds and*:

(A_6) *There exist $\alpha \in L^\infty(0, T; R^+)$ with $\alpha(t) \leqslant \omega^2$ for a.e. $t \in [0, T]$ and*

$$\mathrm{meas}\bigl\{t \in [0, T] \bigm| \alpha(t) < \omega^2\bigr\} > 0,$$

and $\gamma \in L^1(0, T; R^+)$ such that

$$F(t, x) \leqslant \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{14}$$

for all $x \in R^N$ and a.e. $t \in [0, T]$.

Then problem (3) *has at least one solution in H_T^1.*

Remark 3.2. Corollary 3.1 also generalizes Theorem D, so does Theorem C. In fact, there are functions F satisfying our Corollary 3.1 and not satisfying Theorems D and C. For example, let

$$F(t, x) = \frac{1}{2}\beta(t)|x|^2 + \bigl(l(t), x\bigr),$$

where $\beta \in L^\infty(0, T; R^+)$ with $\beta(t) \leqslant \omega^2$ for a.e. $t \in [0, T]$, $\int_0^T \beta(t)\,dt > 0$ and

$$\mathrm{meas}\bigl\{t \in [0, T] \bigm| \beta(t) < \omega^2\bigr\} > 0,$$

and $l \in L^\infty(0, T; R^N)$ with $|l(t)| \leqslant \frac{1}{2}(\omega^2 - \beta(t))$ for a.e. $t \in [0, T]$. Then one has

$$\begin{aligned} F(t, x) &\leqslant \frac{1}{2}\beta(t)|x|^2 + \bigl|l(t)\bigr||x| \\ &\leqslant \frac{1}{2}\bigl(\beta(t) + \bigl|l(t)\bigr|\bigr)|x|^2 + \frac{1}{2}\bigl|l(t)\bigr| \end{aligned}$$

which is just (14) with $\alpha = \beta(t) + |l(t)|$ and $\gamma = \frac{1}{2}|l(t)|$. Hence F satisfies our Corollary 3.1. But in the case that $\mathrm{meas}\{t \in [0, T] \mid \beta(t) = \omega^2\} > 0$, F does not satisfy the conditions of Theorem D or Theorem C.

Theorem 3.2. *Suppose that assumption* (A) *holds and $F(t, x)$ is convex in x for a.e. $t \in [0, T]$. Assume that* (A_3) *holds and*:

(A_7) *There exist $\alpha \in L^\infty(0, T; R^+)$ with $\alpha(t) \leqslant \omega^2$ for a.e. $t \in [0, T]$ and*

$$\mathrm{meas}\bigl\{t \in [0, T] \bigm| \alpha(t) < \omega^2\bigr\} > 0,$$

such that

$$\limsup_{|x|\to\infty} |x|^{-2} F(t,x) \leqslant \frac{1}{2}\alpha(t)$$

uniformly for a.e. $t \in [0, T]$.

Then problem (3) *has at least one solution in* H_T^1.

Remark 3.3. Theorem 3.2 generalizes Theorems D and C. In fact, there are functions F satisfying our Theorem 3.2 and not satisfying Theorem 1.2. For example, let

$$F(t,x) = \frac{1}{2}\alpha(t)|x|^2 + |x|^{\frac{3}{2}} + \bigl(l(t), x\bigr),$$

where $\alpha \in L^\infty(0,T;R^+)$ with $\alpha(t) \leqslant \omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \alpha(t)\,dt > 0$ and

$$\operatorname{meas}\bigl\{t \in [0,T] \bigm| \alpha(t) < \omega^2\bigr\} > 0,$$

and $l \in L^\infty(0,T;R^N)$. Then F satisfies the conditions of our Theorem 3.2. But obviously F does not satisfy the conditions of Theorems D, C and 3.1.

Theorem 3.3. *Suppose that assumption* (A) *holds and* $F(t,x)$ *is convex in* x *for a.e.* $t \in [0,T]$. *Assume that* (A_3) *holds and*:

(A_8) *There exist* $\alpha \in L^1(0,T;R^+)$ *with* $\int_0^T \alpha(t)\,dt < 12/T$ *and* $\gamma \in L^1(0,T;R^+)$ *such that*

$$F(t,x) \leqslant \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{15}$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$.

Then problem (3) *has at least one solution in* H_T^1.

Remark 3.4. There are functions F satisfying our Theorem 3.3 and not satisfying the results mentioned above. For example, let

$$F(t,x) = \frac{1}{2}\beta(t)|x|^2 + \bigl(l(t), x\bigr),$$

where $\beta \in L^1(0,T;R^+)$ with $0 < \int_0^T \beta(t)\,dt < 12/T$ and $l \in L^2(0,T;R^N)$. Then one has

$$\begin{aligned} F(t,x) &\leqslant \frac{1}{2}\beta(t)|x|^2 + \bigl|l(t)\bigr||x| \\ &\leqslant \frac{1}{2}\left(\beta(t) + \frac{12 - T\|\beta\|_{L^1}}{2T^2}\right)|x|^2 + \frac{T^2}{12 - T\|\beta\|_{L^1}}\bigl|l(t)\bigr|^2 \end{aligned}$$

which is just (15) with $\alpha = \beta(t) + \frac{12 - T\|\beta\|_{L^1}}{2T^2}$ and $\gamma = \frac{T^2}{12 - T\|\beta\|_{L^1}}|l(t)|^2$. Thus F satisfies the conditions of our Theorem 3.3. But in the case that $\operatorname{meas}\{t \in [0,T] \mid \beta(t) > \omega^2\} > 0$, F does not satisfy the conditions of Theorems 3.1 and 3.2.

Under assumption (A), the energy functional associated to problem (3), namely φ, on H_T^1 is given by

$$\varphi(u) = -\frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt + \int_0^T F(t, u(t))\,dt$$

which is continuously differentiable, bounded and weakly upper semi-continuous on H_T^1, where

$$H_T^1 = \left\{ u : [0, T] \to R^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; R^N) \end{array} \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot{u}(t)|^2\,dt \right)^{\frac{1}{2}}$$

for $u \in H_T^1$ (see [27]). Moreover, we have

$$\langle \varphi'(u), v \rangle = -\int_0^T (\dot{u}(t), \dot{v}(t))\,dt + \int_0^T (\nabla F(t, u(t)), v(t))\,dt$$

for all $u, v \in H_T^1$ and φ' is weakly continuous. It is well known that weak solutions of problem (3) correspond to the critical points of φ.

For $u \in \widetilde{H}_T^1 \triangleq \{u \in H_T^1 \mid \int_0^T u(t)\,dt = 0\}$, one has Sobolev's inequality

$$\|u\|_\infty^2 \leqslant \frac{T}{12}\|\dot{u}\|_{L^2}^2$$

and Wirtinger's inequality

$$\|u\|_{L^2}^2 \leqslant \frac{T^2}{4\pi^2}\|\dot{u}\|_{L^2}^2$$

(see Proposition 1.3 in [27]).

Lemma 3.1. *Suppose that assumption* (A) *holds. Assume that $F(t, x)$ is convex in x for a.e. $t \in [0, T]$. Then for every $w \in \widetilde{H}_T^1$, $\varphi(x + w)$ is convex in $x \in R^N$.*

Proof. It is obvious that $F(t, x + w(t))$ is convex in $x \in R^N$, so is $\int_0^T F(t, x + w(t))\,dt$. Hence for every $w \in \widetilde{H}_T^1$,

$$\varphi(x + w) = -\int_0^T |\dot{w}(t)|^2\,dt + \int_0^T F(t, x + w(t))\,dt$$

is convex in $x \in R^N$. $\square$

Lemma 3.2. *Suppose that assumptions* (A) *and* (A_3) *hold. Assume that $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Then for every $M > 0$,*

$$\varphi(x+w) \to +\infty$$

as $|x| \to \infty$, $x \in R^N$, uniformly for $w \in \widetilde{H}_T^1$ with $\|w\| \leqslant M$.

Proof. By the convexity of $F(t,\cdot)$, assumption (A) and Sobolev's inequality, we have

$$\begin{aligned}
\int_0^T F(t,x+w)\,dt &\geqslant 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \int_0^T F(t,-w)\,dt \\
&\geqslant 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \int_0^T a\big(\big|w(t)\big|\big)b(t)\,dt \\
&\geqslant 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \max_{0\leqslant s\leqslant \|w\|_\infty} a(s)\int_0^T b(t)\,dt \\
&\geqslant 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \max_{0\leqslant s\leqslant \frac{\sqrt{3T}}{6}\|w\|} a(s)\int_0^T b(t)\,dt \\
&\geqslant 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \max_{0\leqslant s\leqslant \frac{\sqrt{3T}M}{6}} a(s)\int_0^T b(t)\,dt
\end{aligned}$$

for all $x \in R^N$ and $w \in \widetilde{H}_T^1$ with $\|w\| \leqslant M$, which implies that

$$\begin{aligned}
\varphi(x+w) &\geqslant -\frac{1}{2}\|\dot{w}\|^2 + 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \max_{0\leqslant s\leqslant \frac{\sqrt{3T}M}{6}} a(s)\int_0^T b(t)\,dt \\
&\geqslant -\frac{1}{2}M^2 + 2\int_0^T F\left(t,\frac{1}{2}x\right)dt - \max_{0\leqslant s\leqslant \frac{\sqrt{3T}M}{6}} a(s)\int_0^T b(t)\,dt
\end{aligned}$$

for all $x \in R^N$ and $w \in \widetilde{H}_T^1$ with $\|w\| \leqslant M$. Now this lemma follows from (A_3). □

Proof of Theorem 3.1. By Lemmas 3.1 and 3.2 and Theorem 1.1, we only need to prove

$$\varphi(w) \to -\infty$$

as $\|w\| \to \infty \in \widetilde{H}_T^1$. We prove this assertion by contradiction. If not, there exist a constant C_0 and a sequence u_n in $\widetilde{H}_T^1$ such that $\|u_n\| \to \infty$ as $n \to \infty$, and

$$\varphi(u_n) \geqslant C_0 \tag{16}$$

for all n. Hence one has

$$\begin{aligned} C_0 &\leqslant \varphi(u_n) \\ &\leqslant -\frac{1}{2}\int_0^T |\dot{u}_n|^2\,dt + \frac{1}{2}\omega^2\int_0^T |u_n|^2\,dt + \int_0^T \gamma(t)\,dt \\ &\leqslant -\frac{1}{2}\left(1-\frac{1}{4}\right)\|\dot{w}_n\|_{L^2}^2 + \int_0^T \gamma(t)\,dt \end{aligned}$$

which implies that w_n is bounded, where

$$w_n = u_n - a_n\|u_n\|\cos\omega t + b_n\|u_n\|\sin\omega t,$$
$$a_n\|u_n\| = \sum_{i=1}^N (u_n, e_i\cos\omega t)e_i, \qquad b_n\|u_n\| = \sum_{i=1}^N (u_n, e_i\sin\omega t)e_i.$$

Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. It is obvious that $\{a_n\}$ and $\{b_n\}$ are bounded. Hence $\{a_n\}$ and $\{b_n\}$ have a subsequence, denoted by $\{a_n\}$ and $\{b_n\}$ too, which converges $a, b \in R^N$, respectively. By the boundedness of w_n, we have $w_n/\|u_n\| \to 0$. Hence $v_n \to a\cos\omega t + b\sin\omega t$ and $|a| + |b| \neq 0$, which implies that $v_n(t) \to a\cos\omega t + b\sin\omega t$ uniformly for $t \in [0, T]$ by Sobolev inequality. Because that $a\cos\omega t + b\sin\omega t$ has only finite zeros, $|u_n(t)|$ converges to $+\infty$ for a.e. $t \in [0, T]$. Let

$$E = \left\{ t \in [0,T] \;\middle|\; F(t,x) - \frac{1}{2}\omega^2|x|^2 \to -\infty \text{ as } |x| \to \infty \right\}.$$

From Lebesgue–Fatou Lemma (see [54]), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\varphi(u_n) &\leqslant \limsup_{n\to\infty}\int_0^T \left[F(t,u_n) - \frac{1}{2}\omega^2|u_n|^2\right]dt \\ &\leqslant \limsup_{n\to\infty}\int_E \left[F(t,u_n) - \frac{1}{2}\omega^2|u_n|^2\right]dt + \int_0^T \left|\gamma(t)\right|dt \\ &\to -\infty \end{aligned}$$

which contradicts (16).

By Lemmas 3.1 and 3.2 and Theorem 1.1, φ has at least one critical point. Hence problem (3) has at least one solution in H_T^1, which completes our proof. □

Proof of Theorem 3.2. Firstly, there exists a constant $a_0 < 1$ such that

$$\int_0^T \alpha(t)|u|^2\,dt \leqslant a_0\int_0^T |\dot{u}|^2\,dt \tag{17}$$

for all $u \in \widetilde{H}_T^1$. In fact, if not, there exists a sequence $\{u_n\}_{n=1}^\infty \subset \widetilde{H}_T^1$ such that

$$\int_0^T \alpha(t)|u_n|^2\,dt > \left(1-\frac{1}{n}\right)\int_0^T |\dot{u}_n|^2\,dt$$

for all n, which implies that $u_n \neq 0$ for all n. By the homogeneity of the above inequality we may assume that $\int_0^T |\dot{u}_n|^2\,dt = 1$ and

$$\int_0^T \alpha(t)|u_n|^2\,dt > 1-\frac{1}{n} \tag{18}$$

for all n. It follows from the weak compactness of the unit ball of $\widetilde{H}_T^1$ that there exists a subsequence, say $\{u_n\}$, such that u_n weakly converges to u in $\widetilde{H}_T^1$. Hence u_n converges to u in $C(0,T;R^N)$. From (18) we obtain

$$\int_0^T \alpha(t)|u|^2\,dt \geqslant 1.$$

Moreover one has

$$1 \geqslant \int_0^T |\dot{u}|^2\,dt \geqslant \omega^2\int_0^T |u|^2\,dt \geqslant \int_0^T \alpha(t)|u|^2\,dt \geqslant 1.$$

Hence we have

$$1 = \int_0^T |\dot{u}|^2\,dt = \omega^2\int_0^T |u|^2\,dt$$

and

$$\int_0^T \big(\omega^2-\alpha(t)\big)|u|^2\,dt = 0$$

which implies that $u = a\cos\omega t + b\sin\omega t$, $a, b \in R^N$, $u \neq 0$ and $u = 0$ on a positive measure subset. It contradicts the fact that $u = a\cos\omega t + b\sin\omega t$ has finite zeros if $u \neq 0$.

By (A_7) and assumption (A), there exists $M > 0$ such that

$$F(t,x) \leqslant \frac{1}{2}\left(\alpha(t)+\frac{1}{2}(1-a_0)\omega^2\right)|x|^2 + \Big(\max_{0\leqslant s\leqslant M} a(s)\Big)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, which implies that

$$\begin{aligned}\varphi(w) &\leqslant -\frac{1}{2}\|\dot{w}\|_{L^2}^2 + \frac{1}{2}\int_0^T \left(\alpha(t) + \frac{1}{2}(1-a_0)\omega^2\right)|w|^2\,dt + \max_{0\leqslant s\leqslant M} a(s)\int_0^T b(t)\,dt \\ &\leqslant -\frac{1}{2}(1-a_0)\|\dot{w}\|_{L^2}^2 + \frac{1}{4}(1-a_0)\omega^2\int_0^T |w|^2\,dt + \max_{0\leqslant s\leqslant M} a(s)\int_0^T b(t)\,dt \\ &\leqslant -\frac{1}{4}(1-a_0)\|\dot{w}\|_{L^2}^2 + \max_{0\leqslant s\leqslant M} a(s)\int_0^T b(t)\,dt\end{aligned}$$

for all $w \in \widetilde{H}_T^1$ by (17) and Wirtinger's inequality. It follows from the equivalence of the norm $\|\dot{w}\|_{L^2}$ and $\|\cdot\|$ on $\widetilde{H}_T^1$ that

$$\varphi(w) \to -\infty$$

as $\|w\| \to \infty$ in $\widetilde{H}_T^1$. By Lemmas 3.1 and 3.2 and Theorem 1.1, φ has at least one critical point. Hence problem (3) has at least one solution in H_T^1. Theorem 3.2 holds. □

Proof of Theorem 3.3. By (A_8) and Sobolev's inequality, we have

$$\begin{aligned}\varphi(w) &= -\frac{1}{2}\|\dot{w}\|_{L^2}^2 + \int_0^T F(t,w)\,dt \\ &\leqslant -\frac{1}{2}\|\dot{w}\|_{L^2}^2 + \frac{1}{2}\int_0^T \alpha(t)|w|^2\,dt + \int_0^T \gamma(t)\,dt \\ &\leqslant -\frac{1}{2}\|\dot{w}\|_{L^2}^2 + \frac{1}{2}\int_0^T \alpha(t)\,dt \cdot \|w\|_\infty^2 + \|\gamma\|_{L^1} \\ &\leqslant -\frac{1}{2}\|\dot{w}\|_{L^2}^2 + \frac{1}{2}\int_0^T \alpha(t)\,dt \cdot \frac{T}{12}\|\dot{w}\|_{L^2}^2 + \|\gamma\|_{L^1} \\ &= -\frac{1}{2}\left(1 - \frac{T}{12}\int_0^T \alpha(t)\,dt\right)\|\dot{w}\|_{L^2}^2 + \|\gamma\|_{L^1}\end{aligned}$$

for all $w \in \widetilde{H}_T^1$, which implies that

$$\varphi(w) \to -\infty$$

as $\|w\| \to \infty \in \widetilde{H}_T^1$. In a similar way to the proof of Theorem 3.2, Theorem 3.3 holds. □

4. Periodic solutions of second-order nonautonomous subquadratic $\mu(t)$-convex Hamiltonian systems

In this section, we shall give the proof of Theorem 1.3 and some other corresponding results.

Theorem 4.1. *Suppose that assumption* (A) *holds and there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$. *Assume that there exist* $\alpha \in L^\infty(0,T;R^+)$ *with* $\alpha(t) \leqslant \omega^2$ *for a.e.* $t \in [0,T]$ *and*

$$\operatorname{meas}\{t \in [0,T] \mid \alpha(t) < \omega^2\} > 0,$$

and $\gamma \in L^1(0,T;R^+)$ *such that*

$$F(t,x) \leqslant \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{19}$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$. *Then problem* (3) *has at least one solution in* H_T^1.

Remark 4.1. There are functions F satisfying our Theorem 4.1 and not satisfying theorems in the former section. For example, let

$$F(t,x) = \frac{1}{2}\mu(t)|x|^2 + \big(l(t),x\big),$$

where $\mu \in L^1(0,T;R)$ with $\mu^+(t) \leqslant \omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \mu(t)\,dt > 0$ and

$$\operatorname{meas}\{t \in [0,T] \mid \mu^+(t) < \omega^2\} > 0,$$

and $l \in L^\infty(0,T;R^N)$ with $|l(t)| \leqslant \frac{1}{2}(\omega^2 - \mu^+(t))$ for a.e. $t \in [0,T]$. Then one has

$$\begin{aligned} F(t,x) &\leqslant \frac{1}{2}\mu^+(t)|x|^2 + \big|l(t)\big||x| \\ &\leqslant \frac{1}{2}\big(\mu^+(t) + \big|l(t)\big|\big)|x|^2 + \frac{1}{2}\big|l(t)\big| \end{aligned}$$

which is just (19) with $\alpha = \mu^+(t) + |l(t)|$ and $\gamma = \frac{1}{2}|l(t)|$. Hence F satisfies our Theorem 1.2. But in the case that $\operatorname{meas}\{t \in [0,T] \mid \mu(t) < 0\} > 0$, F does not satisfy the conditions of theorems in the former section, for F is not convex in x for $t \in [0,T]$ with $\mu(t) < 0$.

Theorem 4.1 is a straight corollary of the following Theorem 4.2.

Theorem 4.2. *Suppose that assumption* (A) *holds and there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$. *Assume that there exist* $\alpha \in L^\infty(0,T;R^+)$ *with* $\alpha(t) \leqslant \omega^2$ *for a.e.* $t \in [0,T]$ *and*

$$\operatorname{meas}\{t \in [0,T] \mid \alpha(t) < \omega^2\} > 0,$$

such that

$$\limsup_{|x|\to\infty} |x|^{-2} F(t,x) \leqslant \frac{1}{2}\alpha(t) \tag{20}$$

uniformly for a.e. $t \in [0,T]$. *Then problem* (3) *has at least one solution in* H_T^1.

Remark 4.2. There are functions F satisfying our Theorem 4.2 and not satisfying Theorem 4.1. For example, let

$$F(t,x) = \frac{1}{2}\mu(t)|x|^2 + |x|^{\frac{3}{2}} + \big(l(t), x\big),$$

where $\mu \in L^1(0,T;R)$ with $\mu(t) \leqslant \omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \mu(t)\,dt > 0$ and

$$\mathrm{meas}\big\{t \in [0,T] \bigm| \mu(t) < 0\big\} > 0,$$

and $l \in L^\infty(0,T;R^N)$. Then (20) holds with $\alpha = \mu^+(t)$. Hence F satisfies the conditions of our Theorem 4.2. But obviously F does not satisfy the conditions of Theorem 4.1 if $\mathrm{meas}\{t \in [0,T] \mid \mu(t) = \omega^2\} > 0$.

Lemma 4.1. *Suppose that assumption* (A) *holds. Assume that there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt \geqslant 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$. *Then for every* $w \in \widetilde{H}_T^1$, $\varphi(x+w)$ *is convex in* $x \in R^N$.

Proof. Let $G(t,x) = F(t,x) - \frac{1}{2}\mu(t)|x|^2$ for all $x \in R^N$ and a.e. $t \in [0,T]$. Then $G(t, x + w(t))$ is convex in $x \in R^N$, so is $\int_0^T G(t, x + w(t))\,dt$. Hence for every $w \in \widetilde{H}_T^1$,

$$\begin{aligned}
\varphi(x+w) &= -\int_0^T \big|\dot{w}(t)\big|^2\,dt + \frac{1}{2}\int_0^T \mu(t)\big|x + w(t)\big|^2\,dt + \int_0^T G\big(t, x + w(t)\big)\,dt \\
&= -\int_0^T \big|\dot{w}(t)\big|^2\,dt + \frac{1}{2}\int_0^T \mu(t)\,dt\,|x|^2 + \left(\int_0^T \mu(t) w(t)\,dt, x\right) \\
&\quad + \frac{1}{2}\int_0^T \mu(t)\big|w(t)\big|^2\,dt + \int_0^T G\big(t, x + w(t)\big)\,dt
\end{aligned}$$

is convex in $x \in R^N$. $\square$

Lemma 4.2. *Suppose that assumption* (A) *holds. Assume that there exists* $\mu \in L^1(0,T;R)$ *with* $\int_0^T \mu(t)\,dt > 0$ *such that* $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$. *Then for every* $M > 0$,

$$\varphi(x+w) \to +\infty$$

as $|x| \to \infty$, $x \in R^N$, *uniformly for* $w \in \widetilde{H}_T^1$ *with* $\|w\| \leqslant M$.

Proof. Let $G(t,x) = F(t,x) - \frac{1}{2}\mu(t)|x|^2$ for all $x \in R^N$ and a.e. $t \in [0,T]$. By the definition of subdifferential of convex function, we have

$$
\begin{aligned}
F(t,x) - \frac{1}{2}\mu(t)|x|^2 &= G(t,x) \\
&\geqslant G(t,0) + \big(\nabla G(t,0), x\big) \\
&= F(t,0) + \big(\nabla F(t,0), x\big) \\
&\geqslant -a(0)b(t)\big(1+|x|\big)
\end{aligned}
$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. It follows from assumption (A) and Sobolev's inequality that

$$
\begin{aligned}
\varphi(x+w) &\geqslant -\frac{1}{2}\|\dot{w}\|^2 + \frac{1}{2}\int_0^T \mu(t)|x+w|^2\,dt - \int_0^T a(0)b(t)\big(1+|x+w|\big)\,dt \\
&\geqslant -\frac{1}{2}M^2 + \frac{1}{2}|x|^2\int_0^T \mu(t)\,dt - \|w\|_\infty\|\mu\|_{L^1}|x| - \frac{1}{2}\|w\|_\infty^2\|\mu\|_{L^1} \\
&\quad - \big(1+\|w\|_\infty+|x|\big)a(0)\int_0^T b(t)\,dt \\
&\geqslant -\frac{1}{2}M^2 + \frac{1}{2}|x|^2\int_0^T \mu(t)\,dt - \frac{\sqrt{3T}\|w\|}{6}\|\mu\|_{L^1}|x| - \frac{T\|w\|^2}{24}\|\mu\|_{L^1} \\
&\quad - \left(1+\frac{\sqrt{3T}\|w\|}{6}+|x|\right)a(0)\int_0^T b(t)\,dt \\
&\geqslant -\frac{1}{2}M^2 + \frac{1}{2}|x|^2\int_0^T \mu(t)\,dt - \frac{\sqrt{3T}M}{6}\|\mu\|_{L^1}|x| - \frac{T\|w\|^2}{24}\|\mu\|_{L^1} \\
&\quad - \left(1+\frac{\sqrt{3T}M}{6}+|x|\right)a(0)\int_0^T b(t)\,dt
\end{aligned}
$$

for all $x \in R^N$ and $w \in \widetilde{H}_T^1$ with $\|w\| \leqslant M$. Now this lemma follows from $\int_0^T \mu(t)\,dt > 0$. □

Proof of Theorems 4.2 and 1.3. Replacing Lemmas 3.1 and 3.2 in the proofs of Theorems 3.2 and 3.3 by Lemmas 4.1 and 4.2 one can complete the proof of Theorems 4.2 and 1.3. □

5. Periodic solutions of second-order nonautonomous subquadratic $k(t)$-concave Hamiltonian systems

In the last section, we shall give the proof of Theorem 1.4 and some other corresponding results.

Theorem 5.1. *Suppose that assumption* (A) *holds and there exists $k \in L^1(0,T;R^+)$ with $k(t) \leqslant \omega^2$ for a.e. $t \in [0,T]$ and*

$$
\operatorname{meas}\big\{t \in [0,T] \;\big|\; k(t) < \omega^2\big\} > 0,
$$

such that $-F(t,x)+\frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t\in[0,T]$. Assume that (A_3) *is satisfied. Then problem* (3) *has at least one solution in H_T^1.*

Remark 5.1. Theorem 5.1 is another generalization of Theorem I. In fact, in the case that $k=0$ Theorem 5.1 is just Theorem I. There are functions F satisfying our Theorem 5.1 and not satisfying Theorems G, H, I or 1.4. For example, let

$$F(t,x)=\frac{1}{2}\mu(t)|x|^2+\big(p(t),x\big)$$

where $\mu\in L^1(0,T;R)$ with $\mu(t)\leqslant\omega^2$ for a.e. $t\in[0,T]$ and

$$0<\int_0^T\mu^-(t)\,dt<\frac{12}{T}<\int_0^T\mu^+(t)\,dt,$$

$\mu^\pm(t)=\max\{\pm\mu(t),0\}$ and $p\in L^1(0,T;R^N)\setminus\{0\}$. Then F satisfies our Theorem 5.1 with $k=\mu^+$. But F does not satisfy the conditions of Theorem 1.4, for $\int_0^T\mu^+(t)\,dt>12/T$. Consequently F does not satisfy the conditions of Theorems G, H or I, for Theorem 1.4 is a generalization of Theorems G, H and I.

Theorem 5.2. *Suppose that assumption* (A) *holds and $-F(t,x)+\frac{1}{2}\omega^2|x|^2$ is convex in x for a.e. $t\in[0,T]$. Assume that* (A_3) *is satisfied and*

$$\int_0^T\left(\frac{1}{2}\omega^2|a\sin\omega t+b\cos\omega t|^2-F(t,a\sin\omega t+b\cos\omega t)\right)dt\to+\infty \tag{21}$$

as $|a|+|b|\to\infty$, $a,b\in R^N$. Then problem (3) *has at least one solution in H_T^1.*

Corollary 5.1. *Suppose that assumption* (A) *holds and $-F(t,x)+\frac{1}{2}\omega^2|x|^2$ is convex in x for a.e. $t\in[0,T]$. Assume that* (A_3) *is satisfied and there exists $\gamma\in L^1(0,T;R^+)$ such that*

$$F(t,x)\leqslant\frac{1}{2}\omega^2|x|^2+\gamma(t) \tag{22}$$

for all $x\in R^N$ and a.e. $t\in[0,T]$, and

$$\operatorname{meas}\left\{t\in[0,T]\;\middle|\;F(t,x)-\frac{1}{2}\omega^2|x|^2\to-\infty\text{ as }|x|\to\infty\right\}>0. \tag{23}$$

Then problem (3) *has at least one solution in H_T^1.*

Lemma 5.1. *Assume that H is a real Hilbert space, $f:H\times H\to R$ is a bilinear functional. Then $g:H\to R$ given by*

$$g(u)=f(u,u),\quad\forall u\in H,$$

is convex if and only if

$$g(u)\geqslant0,\quad\forall u\in H.$$

Proof. By the definition of convex function, g is convex if and only if

$$\begin{aligned} g\big(\lambda u+(1-\lambda)v\big) &= f\big(\lambda u+(1-\lambda)v, \lambda u+(1-\lambda)v\big) \\ &= \lambda^2 f(u,u)+\lambda(1-\lambda)\big(f(u,v)+f(v,u)\big)+(1-\lambda)^2 f(v,v) \\ &\leqslant \lambda g(u)+(1-\lambda)g(v) \\ &= \lambda f(u,u)+(1-\lambda)f(v,v) \end{aligned}$$

for all $u, v \in H$ and $\lambda \in (0,1)$. This is equivalent to

$$f(u,v)+f(v,u) \leqslant f(u,u)+f(v,v)$$

for all $u, v \in H$. That is,

$$f(u-v, u-v) \geqslant 0$$

for all $u, v \in H$. This holds if and only if

$$g(u) \geqslant 0$$

for all $u \in H$. $\square$

Lemma 5.2. *Suppose that assumption* (A) *holds. Assume that there exists* $k \in L^1(0,T;R)$ *with* $\int_0^T k(t)\,dt < 12/T$ *such that* $-F(t,x)+\frac{1}{2}k(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$*. Then for every* $x \in R^N$, $\psi(x+v)$ *is convex in* $v \in \widetilde{H}_T^1$*, where*

$$\psi(u) = \frac{1}{2}\int_0^T \big|\dot{u}(t)\big|^2\,dt - \int_0^T F\big(t,u(t)\big)\,dt$$

for all $u \in H_T^1$.

Proof. Let $G(t,x) = -F(t,x)+\frac{1}{2}k(t)|x|^2$ for all $x \in R^N$ and a.e. $t \in [0,T]$. Then $G(t, x+v(t))$ is convex in $v \in \widetilde{H}_T^1$ for all $x \in R^N$ and a.e. $t \in [0,T]$. Thus $\int_0^T G(t,x+v(t))\,dt$ is convex in $v \in \widetilde{H}_T^1$ for all $x \in R^N$. Hence by Lemma 5.1, for every $x \in R^N$, to prove

$$\begin{aligned} \psi(x+v) &= \frac{1}{2}\int_0^T \big|\dot{v}(t)\big|^2\,dt - \frac{1}{2}\int_0^T k(t)\big|x+v(t)\big|^2\,dt + \int_0^T G\big(t,x+v(t)\big)\,dt \\ &= \frac{1}{2}\int_0^T \big|\dot{v}(t)\big|^2\,dt - \frac{1}{2}\int_0^T k(t)\big|v(t)\big|^2\,dt + \frac{1}{2}\int_0^T k(t)\,dt|x|^2 \\ &\quad + \left(\int_0^T k(t)v(t)\,dt, x\right) + \int_0^T G\big(t,x+v(t)\big)\,dt \end{aligned}$$

is convex in $v \in \widetilde{H}_T^1$, we only need to prove

$$\frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt - \frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt \geqslant 0$$

for $v \in \widetilde{H}_T^1$. But by Sobolev's inequality, we have

$$\begin{aligned}\frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt &\leqslant \frac{1}{2}\|v\|_\infty^2 \int_0^T k(t)\,dt \\ &\leqslant \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt\end{aligned}$$

for $v \in \widetilde{H}_T^1$. Hence the lemma holds. □

Lemma 5.3. *Suppose that assumption* (A) *holds. Assume that there exists $k \in L^1(0, T; R)$ with $\int_0^T k(t)\,dt < 12/T$ such that $-F(t,x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0,T]$. Then for every $M > 0$,*

$$\psi(x+v) \to +\infty$$

as $\|v\| \to \infty$, $v \in \widetilde{H}_T^1$, uniformly for $x \in R^N$ with $|x| \leqslant M$, where ψ is the same as in Lemma 5.2.

Proof. Let $G(t,x) = -F(t,x) + \frac{1}{2}k(t)|x|^2$ for all $x \in R^N$ and a.e. $t \in [0,T]$. Then $G(t,x)$ is convex in $x \in R^N$. Hence, we have

$$\begin{aligned}G(t,x) &\geqslant G(t,0) + \big(\nabla G(t,0), x\big) \\ &= F(t,0) + \big(\nabla F(t,0), x\big) \\ &\geqslant -a(0)b(t)\big(1+|x|\big)\end{aligned}$$

for every $x \in R^N$. Hence, for $x \in R^N$ with $|x| \leqslant M$, one has

$$\begin{aligned}\psi(x+v) &= \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt - \frac{1}{2}\int_0^T k(t)\left|x+v(t)\right|^2 dt + \int_0^T G\big(t, x+v(t)\big)\,dt \\ &= \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt - \frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt + \frac{1}{2}\int_0^T k(t)\,dt|x|^2 \\ &\quad + \left(\int_0^T k(t)v(t)\,dt, x\right) + \int_0^T G\big(t, x+v(t)\big)\,dt \\ &\geqslant \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt - \frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right| M^2\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^T k(t)\,dt\right|\|v\|_\infty M - a(0)\int_0^T b(t)\big|x+v(t)\big|\,dt \\
&\geqslant \frac{1}{2}\int_0^T \big|\dot v(t)\big|^2\,dt - \frac{1}{2}\int_0^T k(t)\big|v(t)\big|^2\,dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right|M^2 \\
&+\left|\int_0^T k(t)\,dt\right|\|v\|_\infty M - a(0)\big(M+\|v\|_\infty\big)\int_0^T b(t)\,dt \\
&\geqslant \frac{1}{2}\int_0^T \big|\dot v(t)\big|^2\,dt - \frac{1}{2}\|v\|_\infty^2\int_0^T k(t)\,dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right|M^2 \\
&+\left|\int_0^T k(t)\,dt\right|\|v\|_\infty M - a(0)\big(M+\|v\|_\infty\big)\int_0^T b(t)\,dt \\
&\geqslant \frac{1}{2}\left(1-\frac{T}{12}\int_0^T k(t)\,dt\right)\int_0^T \big|\dot v(t)\big|^2\,dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right|M^2 \\
&-\frac{T}{12}\left|\int_0^T k(t)\,dt\right|\left(\int_0^T \big|\dot v(t)\big|^2\,dt\right)^{\frac{1}{2}} M \\
&-a(0)\left(M+\frac{T}{12}\left(\int_0^T \big|\dot v(t)\big|^2\,dt\right)^{\frac{1}{2}}\right)\int_0^T b(t)\,dt
\end{aligned}
$$

for $v \in \widetilde{H}_T^1$. Hence the lemma holds. □

Proof of Theorem 1.4. Theorem 1.4 follows from (A3), Lemmas 5.2 and 5.3 and Theorem 1.1 immediately with

$$
\psi(u) = \frac{1}{2}\int_0^T \big|\dot u(t)\big|^2\,dt - \int_0^T F\big(t, u(t)\big)\,dt
$$

for all $u \in H_T^1$. □

Lemma 5.4. *Suppose that assumption* (A) *holds. Assume that there exists $k \in L^1(0, T; R)$ with $k(t) \leqslant \omega^2$ for a.e. $t \in [0, T]$, such that $-F(t, x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$. Then for every $x \in R^N$, $\psi(x + v)$ is convex in $v \in \widetilde{H}_T^1$.*

Proof. By the proof of Lemma 5.2, we only need to prove

$$
\frac{1}{2}\int_0^T \big|\dot v(t)\big|^2\,dt - \frac{1}{2}\int_0^T k(t)\big|v(t)\big|^2\,dt \geqslant 0
$$

for $v \in \widetilde{H}_T^1$. But by Wirtinger inequality, we have

$$\frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt \leqslant \frac{1}{2}\omega^2 \int_0^T \left|v(t)\right|^2 dt$$
$$\leqslant \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt$$

for $v \in \widetilde{H}_T^1$. Hence the lemma holds. □

Lemma 5.5. *Suppose that assumption* (A) *holds. Assume that there exists $k \in L^1(0, T; R^+)$ with $k(t) \leqslant \omega^2$ for a.e. $t \in [0, T]$ and*

$$\operatorname{meas}\left\{t \in [0, T] \,\middle|\, k(t) < \omega^2\right\} > 0, \tag{24}$$

such that $-F(t, x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$. Then for every $M > 0$,

$$\psi(x + v) \to +\infty$$

as $\|v\| \to \infty$, $v \in \widetilde{H}_T^1$, uniformly for $x \in R^N$ with $|x| \leqslant M$.

Proof. It follows from the first part of the proof of Theorem 3.2 that there exists a constant $a_0 < 1$ such that

$$\int_0^T k(t)|u|^2 dt \leqslant a_0 \int_0^T |\dot{u}|^2 dt$$

for all $u \in \widetilde{H}_T^1$. Moreover by the proof of Lemma 5.3 and Sobolev's inequality, we have

$$\psi(x + v) \geqslant \frac{1}{2}\int_0^T \left|\dot{v}(t)\right|^2 dt - \frac{1}{2}\int_0^T k(t)\left|v(t)\right|^2 dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right| M^2$$
$$+ \left|\int_0^T k(t)\,dt\right| \|v\|_\infty M - a(0)\left(M + \|v\|_\infty\right)\int_0^T b(t)\,dt$$
$$\geqslant \frac{1}{2}(1 - a_0)\int_0^T \left|\dot{v}(t)\right|^2 dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right| M^2$$
$$+ \left|\int_0^T k(t)\,dt\right| \|v\|_\infty M - a(0)\left(M + \|v\|_\infty\right)\int_0^T b(t)\,dt$$
$$\geqslant \frac{1}{2}(1 - a_0)\int_0^T \left|\dot{v}(t)\right|^2 dt - \left|\frac{1}{2}\int_0^T k(t)\,dt\right| M^2$$

$$- T/12\left|\int_0^T k(t)\,dt\right|\left(\int_0^T \left|\dot v(t)\right|^2 dt\right)^{\frac{1}{2}} M$$

$$- a(0)\left(M + T/12\left(\int_0^T \left|\dot v(t)\right|^2 dt\right)^{\frac{1}{2}}\right)\int_0^T b(t)\,dt$$

for $v \in \widetilde{H}_T^1$. Hence the lemma holds. □

Proof of Theorem 5.1. In a way similar to the proof of Theorem 1.4, Theorem 5.1 follows from (A3), Lemmas 5.4 and 5.5 and Theorem 1.1 immediately. □

Lemma 5.6. *Suppose that assumption* (A) *holds and* $-F(t,x) + \frac{1}{2}\omega^2|x|^2$ *is convex in* x *for a.e.* $t \in [0,T]$*. Assume that*

$$\int_0^T \left(\frac{1}{2}\omega^2|a\sin\omega t + b\cos\omega t|^2 - F(t, a\sin\omega t + b\cos\omega t)\right) dt \to +\infty \tag{25}$$

as $|a| + |b| \to \infty$, $a, b \in R^N$*. Then for every* $M > 0$,

$$\psi(x + v) \to +\infty$$

as $\|v\| \to \infty$, $v \in \widetilde{H}_T^1$*, uniformly for* $x \in R^N$ *with* $|x| \leqslant M$.

Proof. By contradiction. If the assert does not hold, there exist two positive constants M and C, and two sequences $x_n \in R^N$ and $v_n \in \widetilde{H}_T^1$ with $|x_n| \leqslant M$ and $\|v_n\| \to \infty$ as $n \to \infty$ such that

$$\psi(x_n + v_n) \leqslant C$$

for all $n \in N$. Let

$$H_1 = R^N \oplus \left(\sin\omega t R^N\right) \oplus \left(\cos\omega t R^N\right).$$

Then $H_T^1 = H_1 \oplus H_1^\perp$. For every $v \in \widetilde{H}_T^1$, there exist $a, b \in R^N$, and $w \in H_1^\perp$ such that $v = a\sin\omega t + b\cos\omega t + w$. It is obvious that

$$\int_0^T \left|\dot w(t)\right|^2 dt \geqslant 4\omega^2 \int_0^T \left|w(t)\right|^2 dt$$

for all $w \in H_1^\perp$. Let

$$G(t,x) = -F(t,x) + \frac{1}{2}\omega^2|x|^2$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Then $G(t,x)$ is convex in $x \in R^N$ and by (25), one has

$$\widetilde{G}(a,b) \triangleq \int_0^T G(t, a\sin\omega t + b\cos\omega t)\,dt \to +\infty$$

as $|a| + |b| \to \infty$, $a, b \in R^N$. It follows from the least action principle that $\widetilde{G}(a,b)$ has a minimum point (a_0, b_0). Moreover, one obtains

$$\frac{\partial \widetilde{G}}{\partial a}(a_0, b_0) = 0, \qquad \frac{\partial \widetilde{G}}{\partial b}(a_0, b_0) = 0,$$

that is,

$$\int_0^T \big(\nabla G(t, a_0\sin\omega t + b_0\cos\omega t), \sin\omega t\big)\,dt = 0,$$
$$\int_0^T \big(\nabla G(t, a_0\sin\omega t + b_0\cos\omega t), \cos\omega t\big)\,dt = 0.$$

By the convexity of $G(t,\cdot)$, we have

$$\begin{aligned} G(t, x+v) \geqslant{} & G(t, a_0\sin\omega t + b_0\cos\omega t) \\ & + \big(\nabla G(t, a_0\sin\omega t + b_0\cos\omega t), x + v - a_0\sin\omega t - b_0\cos\omega t\big). \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^T G(t, x+v)\,dt \geqslant{} & \int_0^T G(t, a_0\sin\omega t + b_0\cos\omega t)\,dt \\ & + \int_0^T \big(\nabla G(t, a_0\sin\omega t + b_0\cos\omega t), x + v - a_0\sin\omega t - b_0\cos\omega t\big)\,dt \\ \geqslant{} & \int_0^T G(t, a_0\sin\omega t + b_0\cos\omega t)\,dt + \int_0^T \big(\nabla G(t, a_0\sin\omega t + b_0\cos\omega t), x + w\big)\,dt \\ \geqslant{} & -\int_0^T F(t, a_0\sin\omega t + b_0\cos\omega t)\,dt - \int_0^T \big(\nabla F(t, a_0\sin\omega t + b_0\cos\omega t), x + w\big)\,dt \\ \geqslant{} & -\max_{0\leqslant s\leqslant a_0+b_0} a(s)\int_0^T b(t)\,dt - \max_{0\leqslant s\leqslant a_0+b_0} a(s)\int_0^T b(t)|x+w|\,dt \\ \geqslant{} & -\max_{0\leqslant s\leqslant a_0+b_0} a(s)\int_0^T b(t)\,dt\big(1 + M + \|w\|_\infty\big) \geqslant -C_1\left(1 + \left(\int_0^T \big|\dot{w}(t)\big|^2\,dt\right)^{\frac{1}{2}}\right) \end{aligned}$$

where

$$C_1 = \max_{0 \leqslant s \leqslant a_0 + b_0} a(s) \int_0^T b(t)\,dt(1+M)(1+\sqrt{12/T})$$

is a constant. Rewrite

$$v_n = a_n \sin \omega t + b_n \cos \omega t + w_n$$

where $a_n, b_n \in R^N$ and $w_n \in H_1^{\perp}$. Then one obtains

$$\begin{aligned}
C &\geqslant \psi(x_n + v_n) \\
&= \frac{1}{2}\int_0^T \left|\dot{v}_n(t)\right|^2 dt - \frac{1}{2}\int_0^T \omega^2 \left|x_n + v_n(t)\right|^2 dt + \int_0^T G\big(t, x_n + v_n(t)\big)\,dt \\
&= \frac{1}{2}\int_0^T \left|\dot{v}_n(t)\right|^2 dt - \frac{1}{2}\omega^2 \int_0^T \left|v_n(t)\right|^2 dt - \frac{1}{2}\omega^2 T|x_n|^2 + \int_0^T G\big(t, x_n + v_n(t)\big)\,dt \\
&= \frac{1}{2}\int_0^T \left|\dot{w}_n(t)\right|^2 dt - \frac{1}{2}\omega^2 \int_0^T \left|w_n(t)\right|^2 dt - \frac{1}{2}\omega^2 T|x_n|^2 + \int_0^T G\big(t, x_n + v_n(t)\big)\,dt \\
&\geqslant \frac{3}{8}\int_0^T \left|\dot{w}_n(t)\right|^2 dt - \frac{1}{2}\omega^2 T M^2 - C_1\left(1 + \left(\int_0^T \left|\dot{w}_n(t)\right|^2 dt\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Hence $w_n \in \widetilde{H}_T^1$ is bounded. Thus, there exists a constant C_3 such that

$$\|w_n\| \leqslant C_3.$$

It follows that

$$\begin{aligned}
C &\geqslant \psi(x_n + v_n) \\
&= \frac{1}{2}\int_0^T \left|\dot{w}_n(t)\right|^2 dt - \frac{1}{2}\omega^2 \int_0^T \left|w_n(t)\right|^2 dt - \frac{1}{2}\omega^2 T|x_n|^2 \\
&\quad + \int_0^T G(t, x_n + a_n \sin \omega t + b_n \cos \omega t + w_n)\,dt \\
&\geqslant -C_4 + 2\int_0^T G\left(t, \frac{1}{2}(a_n \sin \omega t + b_n \cos \omega t)\right) dt - \int_0^T G(t, -x_n - w_n)\,dt.
\end{aligned}$$

Hence the sequences a_n and b_n are bounded, which contradicts the fact that $\|v_n\| \to \infty$. So the lemma holds. □

Proof of Theorem 5.2. In a way similar to the proof of Theorem 1.4, Theorem 5.2 follows from (A3), Lemmas 5.4 and 5.6 and Theorem 1.1 immediately. □

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